Tracing a linearly moving node from asynchronous time-of-arrival measurements

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Abstract—We address the problem of estimating the position and velocity of a radio transmitter moving with constant (unknown) velocity from packet arrival timestamps collected by a set of anchor nodes in fixed known positions. The considered system is completely asynchronous: no assumption is made about node clock synchronization nor about timing of transmitted packets. A distinguishing feature of the proposed model is that it relies exclusively on reception timestamps, with no need to measure nor control transmission times. Because of that, transmitters do not need to cooperate to the tracing process, enabling the opportunistic exploitation of packets that were generated for communication (not localisation) purposes. We consider a batch processing approach, where all the measurements collected within a given observation window are jointly processed. Different Generalized Least Squares formulations are provided for the problem at hand and their equivalence is proved.

I. INTRODUCTION

We consider a wireless system with a blind node transmitting a sequence of data packets at arbitrary (unknown) transmission times while moving linearly at constant velocity. In parallel, a set of fixed anchor nodes in known positions emit beacon packets. Both types of packets are overheard and timestamped by all fixed anchor nodes, and the problem is to determine the (initial or final) position and the velocity vector of the blind node from the timestamps recorded during a given measurement interval. A batch processing approach is considered, where all measurement collected within the observation window are jointly processed in a single run to estimate ex post the trajectory parameters (initial position and velocity) with no prior knowledge about the initial node state. We refer to this problem with the term “tracing” in order to distinguish it from the “tracking” problem, where the instantaneous position of the node is iteratively updated upon arrival of each new measurement, possibly starting from a known (at least approximately) initial state. The tracing problem can be considered as the generalization to a moving node of the classical localization problem for a static node.

The considered system is completely asynchronous and is characterised by the following features:

(a) Nodes are asynchronous: each node clock is affected by unknown time offset and frequency offset (clock skew).
(b) Packets are asynchronous: no explicit control is exerted on the transmission times.
(c) Transmission times are unknown: the tracing process relies exclusively on reception timestamps.
(d) The blind node moves linearly with unknown velocity. We highlight that, owing to (b) and (c), transmitters are not required to cooperate explicitly with the tracing process. This is a key aspect of the considered system, as it allows to exploit “opportunistically” packets transmitted for communication, and not for localisation purposes. A minimum level of cooperation is assumed solely between the receiving (fixed) nodes, for the purpose of sharing the collected measurements, while the blind node does not participate to (and might be unaware of) the tracing process.

Relation to existing literature To the best of our knowledge, there is no solution in the literature for the considered scenario, since all previously proposed methods fail to meet one or more of the system requirements (a)-(d). Most of existing papers referring to “asynchronous localisation” actually rely on tightly controlled and/or measured transmission times, and therefore require full cooperation by the transmitters. This applies to all solutions based on two-way ranging, including e.g. [1], [2], [3], [4] (additional references can be found in [5]), that require tight coordination between node pairs, a new protocol for ranging, and cooperation by all transmitters. In [6], [7] the authors have considered node clock with zero frequency offset, effectively assuming “quasi-synchronous” scenario instead of “fully asynchronous” (following the taxonomy proposed in [4]) and static blind nodes, thus missing requirements (a) and (d). The only previous work considering all three conditions (a)-(c) are [5] and [8]. However, the resolution algorithms proposed in both papers were designed specifically for a static node (missing (d)) and rely on the preliminary estimation of range differences between the blind node and the anchors. Therefore, they cannot be extended directly to the case of a moving node, where blind-anchor distances vary for each measurement.

In summary, the considered system model and the proposed estimation procedure represent the main novel contribution of this letter. For such a system, we derive three different variants of (non-linear) Least Squares (LS) formulations, proving their equivalence.

II. SYSTEM MODEL

A. Notation and assumptions

Consider a system with \( N + 1 \) nodes indexed with \( n = 0, \ldots, N \). Index \( n = 0 \) is reserved for the (moving) blind node, whose trajectory parameters we wish to estimate, whereas the remaining \( N \) (anchor) nodes are placed in fixed known positions. We assume a minimum of five anchors, i.e. \( N \geq 5 \). During the observation interval, the blind node transmits \( M \) (data) packets, while every anchor transmits \( B \) (beacon) packets. All packets from the blind node and from the anchors are overheard and timestamped by all other anchors. The latter
share their measurements with a central entity in charge of the computation. The following notation is introduced:

- \( i(m) \in \{0, \ldots, N\} \) denotes the transmitting node index for the generic \( m \)th packet.
- \( t_m \), the transmission time (according to an absolute reference clock) of the \( m \)th packet transmitted by node \( i(m) \).
- \( R_{nm} \), the reception timestamp of the \( m \)th packet at the receiving node \( n \) as measured by its local clock.
- \( \theta_n \), the clock offset term for node \( n \).
- \( \gamma_n \), the clock skew factor for node \( n \).
- \( p_n \triangleq [x_n, y_n, z_n] \) the position of generic anchor node \( n = 1, \ldots, N \) in the reference 3D coordinate system.
- \( p(t) = p_0 + v \cdot t \) the instantaneous position of the blind node at time \( t \).
- \( v \triangleq [v_x, v_y, v_z] \) the velocity vector of the blind node in the reference 3D coordinate system.
- \( d_{k,n} \triangleq \|p_k - p_n\| \) the Euclidean distances between nodes \( k \) and \( n \).
- \( e_{nm} \) a random error term on the reception timestamp.

Assume that the clock of the generic node \( n \) is affected by two systematic error terms: a temporal offset term \( \theta_n \) and a skew term \( \gamma_n \) (equivalently: a relative frequency offset \( 1 - \gamma_n \)). Furthermore, timestamps are affected by i.i.d. random measurement errors. We consider a simplified scenario where signal propagation occurs through direct Line-of-Sight (LOS) path between any pair of nodes. The 3D position of all anchor nodes is known exactly without error. We assume the blind node position to be constrained on the horizontal plane (hence \( v_z = 0 \)) at known height \( z_0 \), and the problem is to determine the horizontal components \( [x_0, y_0, v_x, v_y] \). With the above notation, the reception timestamp measured at anchor \( n \) for the \( m \)th packet sent by the blind node writes as (blind-to-anchor equation):

\[
\gamma_n \cdot R_{nm} = t_m + \frac{1}{c} \|p_0 + v \cdot t_m - p_n\| + \theta_n + e_{nm} , \tag{1}
\]

where \( c \) denotes the speed of light. If instead the \( m \)th packet is transmitted by anchor \( i(m) \), the reception timestamp at another node \( n \neq i(m) \) writes as (anchor-to-anchor equation):

\[
\gamma_n \cdot R_{nm} = t_m + \frac{d_{i(m),n}}{c} + \theta_n + e_{nm} , \tag{2}
\]

where \( d_{i(m),n} \triangleq \|p_{i(m)} - p_n\| \) denotes the (known) distance between the transmitting and receiving anchor.

### B. Estimation of clock error terms (synchronisation)

We devise a two-stage approach where synchronisation and tracing are performed by two distinct routines: first, clock error terms \( \hat{\theta}, \hat{\gamma} \) are estimated from anchor-anchor measurements, then \( p_0, v \) are determined from blind-to-anchor measurements considering \( \hat{\theta}, \hat{\gamma} \) as known parameters. For the first stage, we follow a procedure similar to the synchronisation method proposed earlier in [1] for a different system model, with the necessary adaptation to suit our scenario. The clock estimation problem can be cast into a linear LS problem:

\[
\hat{\gamma}, \hat{\theta} = \arg \min_{\gamma, \theta} \sum_{m=1}^{B} \sum_{n=1}^{N} \left( \gamma_n R_{nm} - t_m - \frac{d_{i(m),n}}{c} - \theta_n \right)^2 \tag{3}
\]

where the distances between transmitting and receiving anchor nodes \( d_{i(m),n} \) are known. Careful analysis reveals that eq. (3) is underdetermined, i.e., it does not have a unique solution (see discussion in [1, Sec. VI]). To remove the ambiguity, it is sufficient to pick one anchor to serve as clock reference (e.g., \( n = 1 \)), setting \( \hat{\theta}_1 = 0 \) and \( \hat{\gamma}_1 = 1 \) as fixed parameters in (3), then to solve for the remaining \( 2(N - 1) \) clock error variables. After estimating the clock error terms \( \hat{\theta}_n, \hat{\gamma}_n \) from anchor-to-anchor measurements, all reception timestamps for blind-to-anchor packets can be corrected. The generic adjusted timestamp will be denoted by \( R'_{nm} = \hat{\gamma}_n \cdot R_{nm} - \hat{\theta}_n \). This procedure is equivalent to align all anchor clocks to the reference anchor, leaving only an (unknown) offset between the common clock frequency (equal for all anchors after timestamp correction) and the nominal clock frequency. The impact of such a residual offset is however negligible in practice.

### C. Estimation of trajectory parameters (tracing)

Replacing the raw timestamps with the adjusted timestamps, the blind-to-anchor equation (1) rewrites as:

\[
R'_{nm} = t_m + \frac{1}{c} \|p_0 + v \cdot t_m - p_n\| + e_{nm} . \tag{4}
\]

Based on this model equation, we can cast the tracing problem into a Non-linear LS form. We shall consider three different variants.

**Time-of-Flight (TOF):** the transmission times \( t \) are known exactly without error. Together with the measured reception times, this is equivalent to measure the elapsed time between transmission and reception, i.e., the “time-of-flight”. The estimation problem is then formulated as:

\[
\hat{p}, \hat{v}, \hat{t} = \arg \min_{p, v, t} \sum_{m=1}^{N} \left( \frac{\|p_0 + v \cdot t_m - p_n\|}{c} + t_m - R'_{nm} \right)^2 . \tag{5}
\]

The estimator (5) is not applicable in our system, since transmission timestamps remain unknown. Hence the solution to (5) will serve here merely as a reference bound.

**Time-of-Arrival (TOA):** transmission times \( t \) are treated as unknown variables to be estimated jointly with blind node position and velocity, formally:

\[
\hat{p}, \hat{v}, \hat{t} = \arg \min_{p, v, t} \sum_{m=1}^{N} \left( \frac{\|p_0 + v \cdot t_m - p_n\|}{c} + t_m - R'_{nm} \right)^2 . \tag{6}
\]

Estimator (6) can be simplified with a negligible impact on the final solution, by approximating the (unknown) transmission times appearing in the distance term with the (known) arrival times at some reference anchor. Picking again the first anchor \( (n = 1) \) as reference and replacing \( t_m \approx R'_{1m} \) in the distance term, the problem (6) rewrites as:

\[
\hat{p}, \hat{v}, \hat{t} = \arg \min_{p, v, t} \sum_{m=1}^{N} \left( \frac{\|p_0 + v \cdot R'_{1m} - p_n\|}{c} + t_m - R'_{nm} \right)^2 . \tag{7}
\]

The impact of this approximation on the final estimate is in the order of \( \frac{1}{c} \) times the distance between the blind node and the reference anchor. For a blind node moving at speed of \( |v| = 10 \text{ m/s} \) (i.e., \( 36 \text{ km/h} \)) at 300 meters distance.
from the reference anchor, this approximation error is in the order of $10^{-5}$ meters and can be safely neglected in practical applications.

**Time-Difference-of-Arrival (TDOA):** The (unknown) transmission time $t_m$ is eliminated by taking the difference of arrival times at different anchors. For a more compact notation let $\xi_{nk}[m] = \frac{\|p_n + v_R_{mn} - p_k\| - \|p_{n} + v_R_{mn} - p_k\|}{\xi_{nm}^2 + R_{nm}^2}$.

As we prove later in Sec. IV, eq. (7) is an instance of "Type I" formulation (see eq. (14)), hence it can be transformed into the following instance of "Type II" formulation (see eq. (15)):

$$\hat{p}, \hat{v} = \arg\min_{p,v} \sum_{m=1}^{M} \left( \sum_{n=2}^{N} \xi_{n1}[m] - \frac{1}{N} \sum_{n=2}^{N} \xi_{n1}[m] \right)^2$$

or, alternatively, into the equivalent "Type III" form (see eq. (15)):

$$\hat{p}, \hat{v} = \arg\min_{p,v} \sum_{m=1}^{M} \sum_{n=1}^{N} \xi_{nk}[m].$$

Generally speaking, while the three estimators (7), (8) and (9) deliver the same solution, owing to their different structures they will yield different implementation and resolution costs, depending on the capabilities of the particular software and/or hardware computation platform. Each of these estimators could be taken as a starting point towards the development of more robust variants, e.g. in the direction of M-estimators [9]. Note also that all the estimators developed above involve only square terms with no mixed products, despite non-diagonal covariance in TDOA. We expect this feature to be advantageous for implementation in distributed privacy-preserving computation schemes — this is part of our ongoing work.

III. NUMERICAL RESULTS

Hereafter, we present Monte Carlo simulation results for a simplified scenario with $N = 5$ anchors in fixed positions at 5 meters height. The blind node lies within a squared area of interest of size $50 \times 50$ meters, at the (known) fixed height of 1.5 meters. For each trial, the node speed is set randomly between 0 and 5 m/s and the initial node position $p_0$ is extracted randomly within the area of interest. The moving direction is random, with the constraint of ensuring that the final point remains within the area of interest. The (relative) frequency offset and temporal offset of each node clock are extracted randomly in the range $[-40, +40]$ ppm and $[0, 100]$ ns, respectively. Measurement errors are i.i.d. Gaussian with zero mean and standard deviation $\sigma_e = 10$ ns. During the observation period of duration, $D = 3$ s, each anchor emits $B = 6$ beacons while the blind node emits $M = 20$ packets.

Fig. 1 plots the estimation error on the final position at the end of the observation period. The leftmost red curve is obtained from TOF (5) assuming exact knowledge of the actual transmission times $t$ and clock error terms $\gamma, \theta$. The next black curve represents the coincident output of three equivalent estimators (7), (8) and (9) in case of ideal knowledge of the actual clock error terms $\gamma, \theta$, while the rightmost blue curve refers to a more realistic scenario, where clock error terms are estimated from anchor-to-anchor packets with the procedure presented in Sec. II-B. In agreement with the result established in Section IV, in each trial the three estimators lead exactly to the same solution. Comparing the two curves, the loss of accuracy caused by imperfect recovery of clock error terms remains below 1 meter. The inset in Fig. 1 plots the estimated speed $|v|$ versus the actual speed $|v|$ for each trial: no evident correlation appears between estimation error and node speed.

IV. APPENDIX: EQUIVALENCE OF THREE LS FORMS

We present here a general result about the equivalence of certain forms of Generalized LS formulations. In so doing, we show that the elimination (by subtraction) of a common additive nuisance term does not change the global solution. Subtraction introduces noise correlations, hence non-diagonal covariance. However, by exploiting the particular symmetry of the covariance matrix, we derive formulations that contain only square terms, wit no mixed products. To the best of our knowledge, these results were not developed explicitly in any previous work.

Consider a vector $x \equiv [x_1, \ldots, x_K]^T$ of $K$ unknown variables, a nuisance parameter $\tau$, a vector of $N \geq K + 1$ observations $\ell \equiv [\ell_1, \ldots, \ell_N]^T$, a vector of zero-mean i.i.d. random noise terms $e \equiv [e_1, \ldots, e_N]^T$ with unitary variance (COV($e$) = $I$) and a set of model equations in the form:

$$\ell_n = f_n(x_1, \ldots, x_K) + \tau + e_n, \quad n = 1, \ldots, N$$

wherein $f_n()$ are differentiable parametric functions, possibly non-linear. Turning to vectorial notation this model rewrites:

$$\ell = f(x) + \tau \cdot 1_{[N-1,1]} + e.$$ (11)

wherein $1_{[N-1,1]} \equiv [1, \ldots, 1]^T$ denotes the all-one column vector of length $N-1$. The goal is to estimate the variables of interest $x$ from the observed data notwithstanding the nuisance parameter $\tau$. The estimation problem can be casted into LS optimization (Type I formulation):

$$\hat{x}, \hat{\tau} = \arg\min_{x,\tau} e^T e = \arg\min_{x,\tau} \sum_{n=1}^{N} (f_n(x) + \tau - \ell_n)^2.$$ (12)
An alternative approach is to transform the problem in order to eliminate upfront the nuisance parameter \( \tau \). To this aim, it is sufficient to consider pair-wise differences of the equations (10). Let the first observation \( \ell_1 \) serve as common reference and subtract it from the remaining \( N - 1 \) observations:

\[
\Delta \ell_n = \ell_n - \ell_1 = f_n(x) - f_1(x) + \xi_n, \quad n = 2, \ldots, N
\]

where \( \xi_n = \ell_n - \ell_1 \) is a compound error term. In vectorial notation, the new error vector \( \xi \triangleq [\xi_2, \ldots, \xi_N]^T \) is a projection of the originary error vector \( e \) by the projection matrix

\[
D = \begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
-1 & 0 & 0 & \cdots & 1
\end{bmatrix}, \quad \text{i.e.} \quad \xi_{[N-1,1]} = D_{[N-1,N]} \cdot e_{[N,1]}
\]

The covariance matrix of \( \xi \) is derived as \( \Sigma = \text{COV}(\xi) = D \cdot \text{COV}(e) \cdot D^T = D D^T \), since we have assumed independent errors (COV(e) = I). From the particular form of matrix \( D \) it follows that the covariance has the following simple structure:

\[
\Sigma = D D^T = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{bmatrix} = I + 1_{[N-1]} - 1
\]

where \( 1_{[N-1]} \) denotes the square matrix of size \((N-1) \times (N-1)\) with all unitary elements. Therefore, the inverse develops as \( \Sigma^{-1} = (I + 1_{[N-1]})^{-1} = I - \frac{1}{2} 1_{[N-1]} \). From the new set of model equations (13) we derive the following Generalized LS estimate (Type II formulation):

\[
\hat{x} = \arg \min_x \xi^T \Sigma^{-1} \xi = \arg \min_x \xi^T \left( I - \frac{1}{N} 1_{N-1} \right) \xi
\]

\[
\begin{align*}
&= \arg \min_x \left( \sum_{n=2}^{N} \xi_n^2 - \frac{1}{N} \sum_{n=2}^{N} \xi_n^2 \right) \\
&= \arg \min_x \left( \sum_{n=2}^{N} (f_n(x) - f_1(x) - \xi_n + \ell_1)^2 \right) \\
&\quad - \frac{1}{N} \left( \sum_{n=2}^{N} (f_n(x) - f_1(x) - \xi_n + \ell_1)^2 \right) .
\end{align*}
\]

(14)

Furthermore, we consider the following LS minimisation (Type III formulation):

\[
\hat{x} = \arg \min_x \sum_{n=1}^{N} \left( f_n(x) - f_m(x) - \xi_n + \ell_m \right)^2 .
\]

(15)

Note that the double sum in (15) involves \( \frac{N(N-1)}{2} \) squared terms, without mixed products. Hereafter we verify that all three formulations (12), (14) and (15) lead to the same solution \( \hat{x} \). Recall that the minimum of a sum of squares is found by setting the gradient to zero, i.e., by zeroing all partial derivatives. For Type I optimization (12), the gradient equations write (\( k = 1, \ldots, K \)):

\[
\begin{align*}
\frac{\partial}{\partial x_k} \sum_{n=1}^{N} (f_n(x) - f_1(x) + \xi_n - \ell_n + \ell_1)^2 &= 0 \\
\frac{\partial}{\partial x_k} \sum_{n=1}^{N} (f_n(x) + \tau - \ell_n + \ell_1)^2 &= 0
\end{align*}
\]

(16)

that, after developing the derivatives and eliminating \( \tau \) from the equations, lead to the following equations (\( k = 1, \ldots, K \)):

\[
\begin{align*}
\sum_{n=1}^{N} (f_n(x) - \ell_n) \frac{\partial f_n(x)}{\partial x_k} &= 0, \\
- \frac{1}{N} \sum_{n=1}^{N} (f_n(x) - \ell_n) \cdot \sum_{n=1}^{N} \frac{\partial f_n(x)}{\partial x_k} &= 0.
\end{align*}
\]

(17)

For Type II the gradient equations are derived from (14):

\[
\begin{align*}
\sum_{n=2}^{N} \left( f_n(x) - \ell_n - f_1(x) + \ell_1 \right) \left( \frac{\partial f_n(x)}{\partial x_k} - \frac{\partial f_1(x)}{\partial x_k} \right) \\
- \frac{1}{N} \sum_{n=2}^{N} \left( f_n(x) - \ell_n - f_1(x) + \ell_1 \right) = 0
\end{align*}
\]

(18)

After some simple but cumbersome algebraic manipulations, condition (18) can be conducted to the same form of (17). Similarly, for Type III (eq. (15)) the gradient equations write:

\[
\begin{align*}
\sum_{n=1}^{N} \sum_{m=1}^{N} \left( f_n(x) - f_m(x) + \ell_m \right) \left( \frac{\partial f_n(x)}{\partial x_k} - \frac{\partial f_m(x)}{\partial x_k} \right) &= 0
\end{align*}
\]

(19)

where the inner sum replacement \( \sum_{m=1}^{M} \rightarrow \sum_{m=n+1}^{M} \) is justified by the symmetry between the terms. Again, with some algebraic manipulations the equations (19) can be conducted to the same form as (17). Therefore we conclude that all three formulations have exactly the same solution \( \hat{x} \).

This is a stronger result than the equivalence of the Cramér-Rao lower bound for TOA and TDOA proved earlier in [10] (for the particular case of static node localisation) and provides a formal explanation for the simulation results presented in [10] but left unexplained therein. Furthermore, our result applies to any differentiable model function \( f(\cdot) \), including non-linear forms, therefore it extends the previous result in [11, Theorem 2] about the equivalence of approximated linearised instances of TOA and TDOA for a static node.

References


